Ch: Set & Seq
Part 5

V. imp

Limit point of a sequence

Defn: A real no. α is said to be a limit pt / cluster pt of a seq $\{x_n\}$ if every ε-nbd of α contains an infinite number of members of $\{x_n\}$.

ie $\forall \varepsilon > 0$, $|x_n - \alpha| < \varepsilon$ for infinitely many n.

Ex: $\{\frac{1}{n}\}$ has range set $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

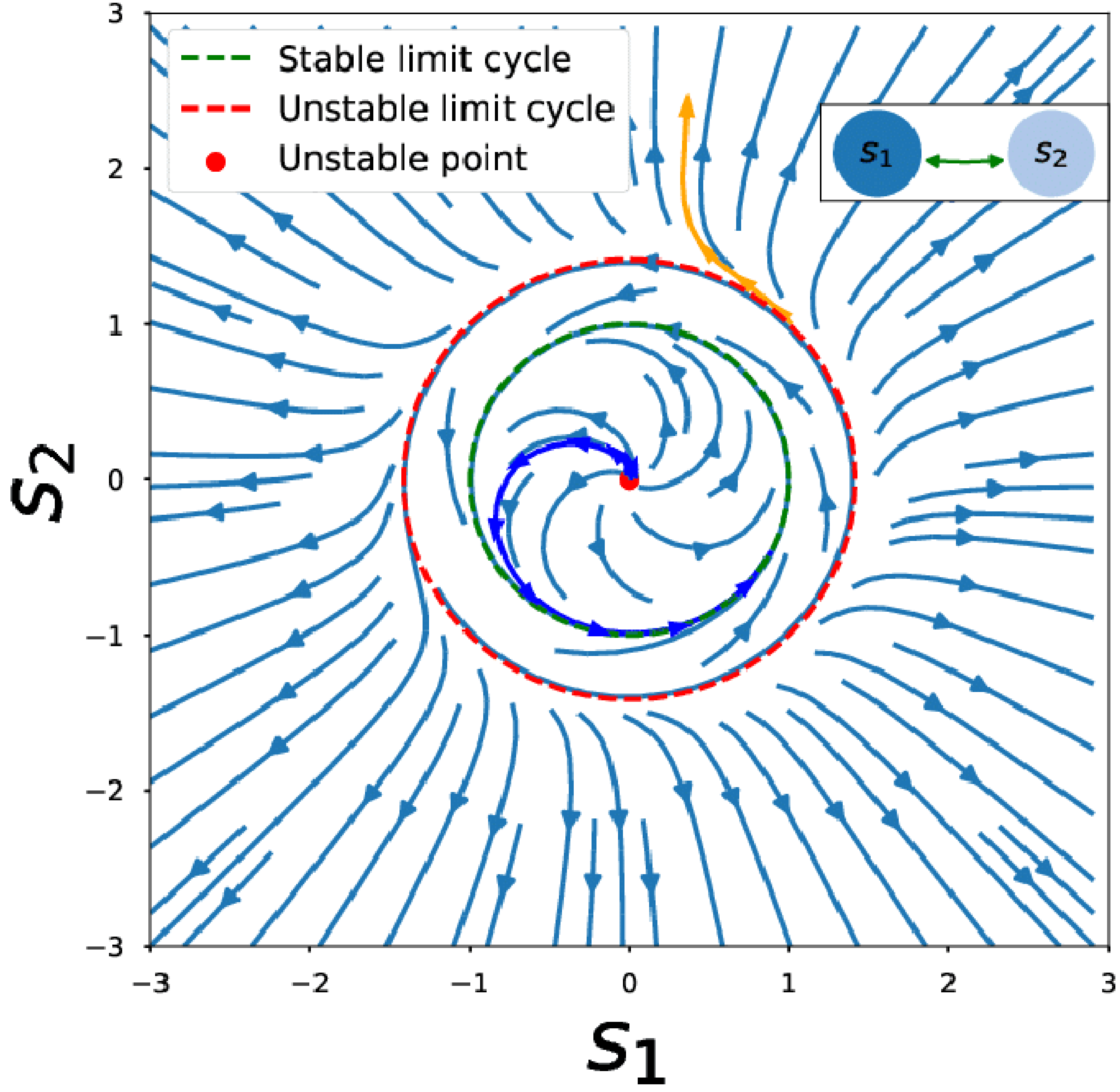
$\longrightarrow$ limit pt $= 0$

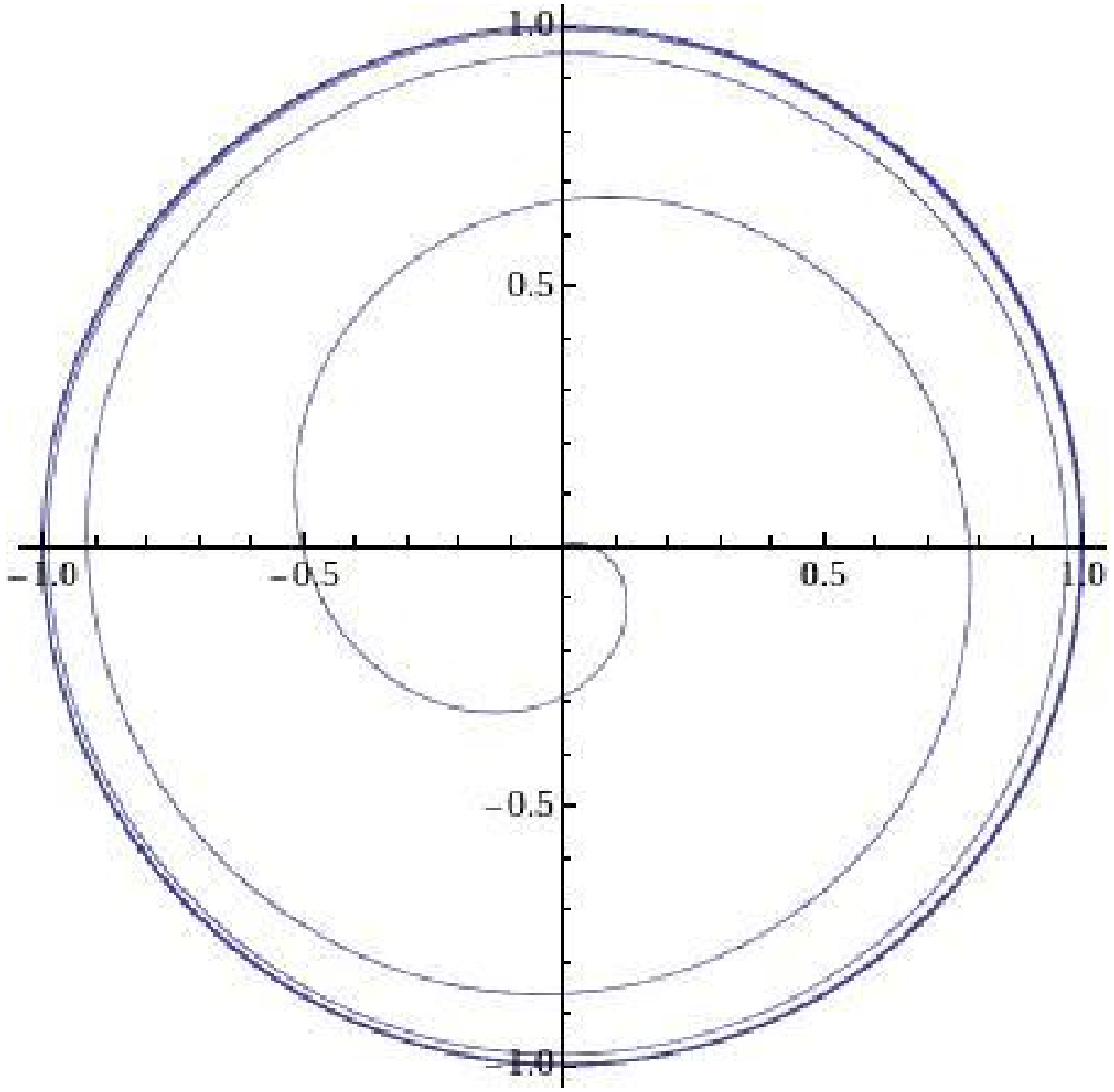
H.W: (1) $\{(-1)^n\}$ has range set $\{-1, 1\}$

both are lim pt $\begin{bmatrix} -1 \text{ for } n = 1,3,5,\dots \\ +1 \text{ for } n = 2,4,6,\dots \end{bmatrix}$

(2) $\{(-1)^n + \frac{1}{n}\}$ $\rightarrow$ find limit pts.

-1, -1





https://demonstrations.wolfram.com/FlowerAroundALimitPoint/

%3A%2F%2Fwww.researchgate.net%2Ffigure%2FIllustration-of-abstract-system-consisting-of-two-variables-s-1-s-2-The-system_fig3_332439183&psig=AOvVaw15S5DPgjrutCqn-jMOBj_m&ust=1632887453119000&source=images&cd=vfe&ved=0CAwQjhxqFwoTCNjiuYLjoP

Bolzano - Weierstrass Theorem

Every bounded seq has a limit point.

Theorem 1: Let $\{x_n\}$ be bdd seq and $\{y_n\} \to 0$

Then $\lim\limits_{n\to\infty} x_n y_n = 0$

Theorem 2: The set of limit points of a bdd seq has a greatest (upper limit) limsup or $\overline{\lim}$ & the least (Lower limit) liminf or $\underline{\lim}$.

v.v.Imp

Limit Superior & Limit Inferior

$\limsup\limits_{n\to\infty} x_n$ or $\overline{\lim\limits_{n\to\infty}}\, x_n$ = greatest limit point of a bdd seq

$\liminf\limits_{n\to\infty} x_n$ or $\underline{\lim\limits_{n\to\infty}}\, x_n$ = smallest limit point of a bdd seq

When $\{x_n\}$ convergent (ie. $\lim x_n$ exist)

then $\overline{\lim\limits_{n\to\infty}}\, x_n = \lim\limits_{n\to\infty} x_n = \underline{\lim\limits_{n\to\infty}}\, x_n$

(1) $\overline{\lim} x_n = l_1 \Rightarrow \exists N_1$ s.t $x_n - l_1 < \varepsilon \ \forall n \geq N_1$

(2) $\underline{\lim} x_n = l_2 \Rightarrow \exists N_2$ s.t $x_n - l_2 > -\varepsilon \ \forall n \geq N_2$

(3) $$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{m\geq n} x_m$$
$$= \inf_n \left(\sup \left(x_n, x_{n+1}, x_{n+2}, \ldots \right) \right)$$

(4) $$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf_{m\geq n} x_m$$
$$= \sup_n \left(\inf \left(x_n, x_{n+1}, x_{n+2}, \ldots \right) \right)$$

Examples

(1) $\{(-1)^n\}$ $\rightarrow$ $\overline{\lim} = \inf_n \left(\sup(-1, 1, -1, 1, \cdots) \right)$

$= \inf_n 1 = 1$

$\rightarrow$ $\underline{\lim} = \sup_n (\inf(1, -1, \cdots)) = -1$

(2) $\{1 + (-1)^n\}$ $\rightarrow$ $\inf_n \left(\sup(0, 2, 0, 2, \cdots) \right) = 2$

$\rightarrow$

(3) $\left\{ \dfrac{(-1)^n}{n} \right\}$ $\rightarrow$ $\inf_n \left(\sup\left(-\frac{1}{n}, +\frac{1}{n+1}, -\frac{1}{n+2}, \cdots \right) \right)$

$= \inf_n \frac{1}{n+1} = 0$

$\rightarrow$

(4) $\left\{ (-1)^n \left(1 + \frac{1}{n}\right) \right\}$

Ans: $1, -1$

H.W- Find $\overline{\lim}$ and $\underline{\lim}$ for following seq:

(1) $\left\{(-1)^n\left(1+\frac{1}{2n}\right)\right\}$	Ans: $1, -1$
(2) $\left\{n+\frac{(-1)^n}{n}\right\}$	Ans: ∞, ∞
(3) $\left\{n^{(-1)^n}\right\}$	Ans: $\infty, 0$
(4) $\left\{\left(\sin\left(\frac{n\pi}{4}\right)\right)^{(-1)^n}\right\}$	Ans: $\sqrt{2}, -\sqrt{2}$

Theorem: $\underline{\lim}_{n\to\infty} x_n \leq \overline{\lim}_{n\to\infty} x_n$

H.W: (Write down the proof, refer to any textbook; eg: Apostol.)

Theorem: for two bdd seq $\{x_n\}$, $\{y_n\}$

(1) $\overline{\lim}\,(x_n + y_n) \leq \overline{\lim}\, x_n + \overline{\lim}\, y_n$

(2) $\underline{\lim}\,(x_n + y_n) \geq \underline{\lim}\, x_n + \underline{\lim}\, y_n$

Subsequence

Let $\{x_n\}$ be a real seq.
$\{r_n\}$ be a strictly increasing seq of natural number.
Then $\{x_{r_n}\}$ is called subsequence of $\{x_n\}$

eg: $\{\frac{1}{2n}\}$ and $\{\frac{1}{n^2}\}$ are subseq of $\{\frac{1}{n}\}$

Theorem : If $\lim_{n \to \infty} x_n = l$ then every subseq of x_n converges to l.

Ex: Show that $\left\{ (-1)^n + \frac{1}{n} \right\}$ is not convergent.

proof : $x_n = (-1)^n + \frac{1}{n}$

$$x_{2n} = 1 + \frac{1}{2n} \Rightarrow x_{2n} \longrightarrow 1$$

$$x_{2n+1} = -1 + \frac{1}{2n} \Rightarrow x_{2n+1} \longrightarrow -1$$

Contradiction, as every subseq has to have same lim

Bolzano-Weierstrass Thm (Subseq)

Every bdd seq has a convergent subseq

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v. Imp

Cauchy Sequence * (always converges)

A seq is said to be cauchy seq if for every $\varepsilon > 0$, $\exists$ a positive int m such that $|x_p - x_q| < \varepsilon \quad \forall p, q \geq m$

Cauchy's general principle of convergence

A necessary & sufficient condition for convergence of a seq $\{x_n\}$ is :

$\forall \varepsilon > 0 \ \exists m \in \mathbb{N} \ni |x_{n+p} - x_n| < \varepsilon \ \forall n \geq m$ and $p \geq 1$

* Helpful for checking convergence without knowing limit

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots = \sum \frac{1}{n}$ is divergent

Ex show that :

But $1+\frac{1}{2}+\frac{1}{2^2}+\cdots = 2$

① $\{\frac{1}{n}\}$ is convergent

for $p \geq 1$

Ans: $x_n = \frac{1}{n}$.

$$|x_{n+p} - x_n| = \left|\frac{1}{n+p} - \frac{1}{n}\right| = \frac{p}{n(n+p)} < \frac{1}{n}$$

Now, for $\varepsilon > 0$, $\frac{1}{n} > \varepsilon$ if $n > \frac{1}{\varepsilon}$ [Take $m = [\frac{1}{\varepsilon}+1]$]

② Show that $x_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ is divergent

Hint: $|x_{n+p} - x_n| = \left|\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+p}\right| > \frac{p}{n+p} \geq \frac{1}{2}$

Take $\varepsilon = 1/3$ and $n = p = m$ (Contradiction)

Ex (1) Show that $x_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}$ is convergent & cauchy seq.

(2) $x_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n}$ is cauchy seq & convergent.

(3) If $\{x_n\}, \{y_n\}$ cauchy then $\{x_n + y_n\}$ also cauchy

(4) check if $\left\{\frac{n-1}{n+1}\right\}$, $\left\{\frac{1}{n^2}\right\}$ cauchy

(5) Show that, $x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$ is convergent.

Ans

$$|x_{n+p} - x_n| = \left| \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+p)^2} \right|$$

(from defn of limit of $\frac{1}{n}$, $< \frac{n}{(n+1)^2} < \frac{1}{n} \downarrow 0$ as $n \to \infty$

$\exists N \in \mathbb{N} \ni |x_{n+p} - x_n| < \varepsilon \ \forall n \geq N$ and $p \geq 1$

$\Rightarrow$ Cauchy $\Rightarrow$ Convergent)